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On solving evolution equations on Lie groups*

Philip Feinsilver†, Uwe Franz‡ and René Schott§

† Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, USA
‡ Institut für Mathematik und Informatik, Ernst-Moritz-Arndt-Universität Greifswald, 17487 Greifswald, Germany
§ IECN and LORIA Université Henri Poincaré, Nancy 1, 54506 Vandoeuvre-lès-Nancy, France

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Abstract. After a review of operator calculus methods for working with Lie algebras and Lie groups, we discuss solving evolution equations associated with Lie groups. Our techniques involve a dual approach to the Wei–Norman method and a formulation of the propagator in terms of corresponding stochastic processes. Solutions with polynomial initial conditions yield generalized Appell systems. The theory is illustrated using the affine group and a family of nilpotent Lie groups.

1. Introduction

Finding explicit solutions of evolution equations is one of the most important problems in mathematical physics. Prominent examples include Schrödinger's equation, the heat/diffusion equation, the Fokker–Planck equation and master equations in statistical mechanics.

Even in relatively simple cases, for equations on Lie groups general solutions are not known explicitly. The operator calculus methods that we have developed provide an approach for computing solutions that is particularly effective for, but not limited to, polynomial initial conditions. These solutions are (generalized) Appell systems [5]. They generalize the well known Hermite and Laguerre polynomials which arise classically as solutions to the heat equation (with a constant diffusion coefficient).

Here, in section 2 we present an updated review of our approach to computing representations. The idea of starting from the universal enveloping algebra is based originally on work of Gruber *et al* (see, e.g., [6]), and we have developed methods for general Lie groups, providing the basis of the theory and applications presented in Feinsilver and Schott [3]. In section 3 the connection between evolution equations and stochastic processes on Lie groups is presented. Section 4 is devoted to generalized Appell systems. Section 5 illustrates Appell systems for some specific Lie groups. We find connections with the theory of Lie symmetries for differential equations.

The method we use, based on the action of the Lie algebra on the universal enveloping algebra, turns out to be dual to the Wei–Norman method, the common ground of the two approaches being the use of coordinates of the second kind, equivalently, factorization of the group elements into one-parameter subgroups each generated by a basis element of the Lie algebra. Cariñena *et al* [1] have developed applications of the Wei–Norman method as a

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method of group-theoretical solution of equations such as the Riccati equation, which in turn has connections with the Schrödinger equation.

An interpretation of section 4 is that we are presenting a formulation of the propagator in terms of stochastic processes. This depends on connecting terms of the Hamiltonian with certain stochastic processes. A path-integral approach very close in spirit to ours has been developed by Tomé [11]. He also uses the dual representations, for example, see his sections 4.1, 4.2, 5.1, etc. The right dual representation plays an essential role in his formulation. He shows some examples using the affine group; we will use the affine group (and some nilpotent groups) here for illustration as well.

The algebraic basis of the approach is emphasized in this paper, with the aim of increasing accessibility to readers and consequently the usefulness of these techniques. We will develop the approach both in scope (e.g. semisimple groups) and generality (connection with Lie symmetries) in future work.

2. Lie algebras, universal enveloping algebra and duality

2.1. Notation

Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{k}$ (in this paper, $\boldsymbol{k}$ is either the real or the complex numbers). $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ and $U(\mathfrak{g}^{\mathrm{op}}) \cong U(\mathfrak{g})^{\mathrm{op}}$ its opposite. Let $S : U(\mathfrak{g}) \to U(\mathfrak{g}^{\mathrm{op}}) \cong U(\mathfrak{g})^{\mathrm{op}}$ be the unique homomorphism determined by $S(X) = -X$ for all $X \in \mathfrak{g}$.

We will use standard multi-index notation, for example, $n! = \prod n_i!$, $x^n = \prod x_i^{n_i}$, for $n = (n_1, \ldots, n_d)$ and $x = (x_1, \ldots, x_d)$. Multi-indices $n, m \in \mathbb{N}^d$ are ordered by $n \geqslant m$ iff $n_i \geqslant m_i$ for all $i = 1, \ldots, d$.

Given a basis $\{\xi_1 \ldots, \xi_d\}$ for $\mathfrak{g}$, we will denote the corresponding basis for $U(\mathfrak{g})$ by

$$[\![n]\!] = [\![n_1 n_2 \ldots n_d]\!] = \xi_1^{n_1} \cdots \xi_d^{n_d}$$

where we modify Dirac's notation $|n\rangle$ since in the algebra we will consider multiplication from both the left and the right. We will use the Dirac notation for the basis of a vector space, representation space, where the action is from the left.

For functions of variables $A = (A_1, \ldots, A_d)$, we write ∂_i for the partial derivative operator $\partial/\partial A_i$. We will thus use $D_i = \partial/\partial x_i$ for differentiation with respect to variables x_i.

Our *summation convention* is as follows: repeated Greek indices are assumed to be summed over, while Latin indices are not summed unless the summation sign is explicit.

In the sections dealing with evolution equations, we often abbreviate $\partial u/\partial t$, e.g., by u_t, as is common.

Angled brackets are used to denote an expected value, e.g. $\langle Y\rangle$ denotes the expected value of the random variable Y.

2.2. Examples

We now introduce the Lie groups and their Lie algebras that we will use for illustration in the following sections.

2.2.1. The affine group and its Lie algebra. First, consider the group of affine motions on the real line $\mathrm{Aff}(1) = \{g(a_1, a_2) \in \mathbb{R}^2; a_1 > 0\}$. Its action on functions and the corresponding group law are given by

$$g(a_1, a_2) f(x) = f(a_1 x + a_2) \tag{1}$$

$$g(a_1, a_2) \circ g(b_1, b_2) = g(a_1 b_1, b_1 a_2 + b_2). \tag{2}$$

Aff(1) is a Lie group with a basis of the tangent space at the unit element $g(1, 0)$ given by

$$\begin{aligned} \xi_1 f(x) &= \frac{\partial}{\partial a_1} \big(g(a_1, a_2) f(x)\big)\big|_{(1,0)} = x \frac{\mathrm{d}f}{\mathrm{d}x} \\ \xi_2 f(x) &= \frac{\partial}{\partial a_2} \big(g(a_1, a_2) f(x)\big)\big|_{(1,0)} = \frac{\mathrm{d}f}{\mathrm{d}x}. \end{aligned} \tag{3}$$

They satisfy the commutation relation $[\xi_2, \xi_1] = \xi_2$. We note that this is the only non-commutative two-dimensional Lie algebra.

2.2.2. N-step nilpotent Lie algebras. First recall the Heisenberg group. One way to define it is via the group action

$$g(a_0, a_1, a_2) f(x) = \mathrm{e}^{h(a_1 x + a_2)} f(x + a_0) \tag{4}$$

with constant h. Then the Lie algebra is given by differentiation with respect to the parameters a_i at 0 as in (3):

$$\xi_0 f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} \qquad \xi_1 f(x) = hx\, f(x) \qquad \xi_2 f(x) = h\, f(x). \tag{5}$$

This can be embedded in a higher-dimensional analogue as follows. Let $\boldsymbol{a}$ denote the parameters $a_0, a_1, \ldots, a_N$ and set

$$p(\boldsymbol{a}; x) = \sum_{i=1}^{N} a_i \frac{x^{N-i}}{(N-i)!}.$$

Define the group via the action on functions

$$g(\boldsymbol{a}) f(x) = \exp[h\, p(\boldsymbol{a}; x)]\, f(x + a_0). \tag{6}$$

The group law is thus

$$g(\boldsymbol{a}) \circ g(\boldsymbol{b}) = g\left(a_0 + b_0, \ldots, a_i + \sum_{j=0}^{i-1} \frac{a_0^j}{j!} b_{i-j}, \ldots, a_N + \sum_{j=0}^{N-1} \frac{a_0^j}{j!} b_{N-j}\right) \tag{7}$$

which follows by expanding $p(\boldsymbol{a}; x) + p(\boldsymbol{b}; x + a_0)$ and collecting terms. We have

$$\xi_0 f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} \qquad \text{and} \qquad \xi_i f(x) = h \frac{x^{N-i}}{(N-i)!} f(x) \tag{8}$$

for $1 \leqslant i \leqslant N$. The Lie algebra $\mathfrak{g}$ has as a basis $\{\xi_0, \xi_1, \ldots, \xi_N\}$, satisfying the commutation relations

$$\begin{aligned} &[\xi_0, \xi_N] = 0 \\ &[\xi_0, \xi_i] = \xi_{i+1} \qquad 1 \leqslant i < N \\ &[\xi_i, \xi_j] = 0 \qquad 1 \leqslant i, \quad j \leqslant N \end{aligned} \tag{9}$$

and is nilpotent. For $N = 3$ this may be identified as the centrally extended Galilean algebra in 1 + 1 dimensions.

2.3. Representations on $U(\mathfrak{g})$

Consider the universal enveloping algebra $U(\mathfrak{g})$. In this algebra, multiplication from the right and from the left give mappings

$$\rho_L,\ \rho_R : U(\mathfrak{g}) \to \mathrm{Hom}\,(U(\mathfrak{g}), U(\mathfrak{g}))$$

in the following way. For $X_1, X_2, X \in U(\mathfrak{g})$, define

$$\rho_L(X_1)X = X_1 X \qquad \rho_R(X_1)X = X X_1$$

which satisfy

$$\rho_L(X_1) \circ \rho_L(X_2) = \rho_L(X_1 X_2) \qquad \rho_R(X_1) \circ \rho_R(X_2) = \rho_R(X_2 X_1)$$

i.e. ρ_L is a representation of $U(\mathfrak{g})$ and ρ_R is a representation of $U(\mathfrak{g})^{\mathrm{op}}$ called the left and right regular representations, respectively. Note that $\rho_R \circ S$ is a representation of $U(\mathfrak{g})$. Further representations of $U(\mathfrak{g})$ can be obtained from ρ_L by taking the quotient with respect to invariant subspaces $\mathcal{I} \subset U(\mathfrak{g})$; in particular, finite-dimensional representations can be obtained if the codimension is finite. A ρ_L-invariant subspace $\mathcal{I} \subset U(\mathfrak{g})$ is a left ideal, i.e. $X_2 \in \mathcal{I}$ and $X_1 \in U(\mathfrak{g})$ implies $\rho_L(X_1)X_2 \in \mathcal{I}$. Given the basis $\{ [\![n]\!] \}$ for $U(\mathfrak{g})$, we can define matrix coefficients $M_{mn}(X)$ of $\rho_L(X)$ with respect to this basis by

$$\rho_L(X)[\![n]\!] = \sum_{m \geqslant 0} M_{mn}(X)[\![m]\!] \tag{10}$$

for $X \in U(g)$ (note that m and n are generally multi-indices). For $X \in U(\mathfrak{g})$, one also has the adjoint representation, a Lie algebra representation, given by $\mathrm{ad}(X)Y = XY - YX$ acting on $Y \in U(\mathfrak{g})$.

Example 2.1. Consider the Lie algebra for the affine group. In the enveloping algebra, the commutation relation $[\xi_2, \xi_1] = \xi_2$ is the same as $\xi_2\xi_1 - \xi_1\xi_2 = \xi_2$ and hence $\xi_1\xi_2 = \xi_2(\xi_1 - 1)$ and $\xi_2\xi_1 = (1 + \xi_1)\xi_2$.

Now, take the basis $\{ [\![n_1 n_2]\!] \} = \{\xi_1^{n_1}\xi_2^{n_2}\}$ of $U(\mathfrak{g})$. Then the matrix coefficients of the right and left action of the basis elements of the Lie algebra can be read from the following relations:

$$\begin{aligned} \rho_L(\xi_1)[\![n_1 n_2]\!] &= \xi_1 \cdot \xi_1^{n_1}\xi_2^{n_2} = \xi_1^{n_1+1}\xi_2^{n_2} \\ &= [\![n_1 + 1, n_2]\!] \end{aligned} \tag{11}$$

$$\begin{aligned} \rho_L(\xi_2)[\![n_1 n_2]\!] &= \xi_2 \cdot \xi_1^{n_1}\xi_2^{n_2} = (1 + \xi_1)^{n_1}\xi_2^{n_2+1} \\ &= \sum_{\nu=0}^{n} \binom{n}{\nu} [\![n_1 - \nu, n_2 + 1]\!] \end{aligned} \tag{12}$$

$$\rho_R(\xi_1)[\![n_1 n_2]\!] = [\![n_1 + 1, n_2]\!] + n_2[\![n_1 n_2]\!] \tag{13}$$

$$\rho_R(\xi_2)[\![n_1 n_2]\!] = [\![n_1, n_2 + 1]\!]. \tag{14}$$

Next, we give an example of a quotient representation. Let $\mathcal{I}$ be the left ideal generated by $\xi_1 - \alpha$, i.e. $\mathcal{I} = \{X \cdot (\xi_1 - \alpha);\ X \in U(\mathfrak{g})\}$. We can formulate this as the action of $U(\mathfrak{g})$ on a vector space $\{X\Omega\colon X \in U(\mathfrak{g})\}$, i.e. formed by the action of $U(\mathfrak{g})$ on a fixed ('vacuum') vector Ω, with the property that $(\xi_1 - \alpha)\Omega = 0$, equivalently, $\xi_1\Omega = \alpha\Omega$. Then the basis reduces to $\{ |m\rangle \} = \{ \xi_2^m \Omega \}$ and we obtain

$$\begin{aligned} \tilde{\rho}(\xi_1)|m\rangle &= \xi_1 \cdot \xi_2^m\Omega = \xi_2^m(\xi_1 - m)\Omega = (\alpha - m)|m\rangle \\ \tilde{\rho}(\xi_2)|m\rangle &= |m + 1\rangle \end{aligned} \tag{15}$$

where the action is only from the left.

2.4. Canonical bosons

It is convenient to represent the action of multiplication by the basis elements ξ_i on $U(\mathfrak{g})$ by using bosons. We introduce raising operators $\mathcal{R}_i$ and lowering ('velocity') operators $\mathcal{V}_i$ acting on a basis by

$$\mathcal{R}_i[\![n]\!] = [\![n_1, \ldots, n_i + 1, \ldots, n_d]\!] \tag{16}$$

$$\mathcal{V}_i[\![n]\!] = n_i\,[\![n_1, \ldots, n_i - 1, \ldots, n_d]\!]. \tag{17}$$

These satisfy the canonical commutation relations $[\mathcal{V}_j, \mathcal{R}_i] = \delta_{ij} I$, where I is the identity operator. However, note that there is no canonical inner product implied, so that, for example, $\mathcal{R}_i$ and $\mathcal{V}_i$ are not adjoint to each other. This is why we are not employing the traditional $a, a^\dagger$ notation.

Example 2.2. We continue with the affine algebra. The results (11)–(14) can be conveniently reformulated as

$$\rho_L(\xi_1) = \mathcal{R}_1 \tag{18}$$

$$\rho_L(\xi_2) = \mathcal{R}_2 \exp(\mathcal{V}_1) \tag{19}$$

$$\rho_R(\xi_1) = \mathcal{R}_1 + \mathcal{R}_2\mathcal{V}_2 \tag{20}$$

$$\rho_R(\xi_2) = \mathcal{R}_2. \tag{21}$$

The quotient representation, equation (15), can be written as

$$\tilde{\rho}(\xi_1) = \alpha I - \mathcal{R}\mathcal{V} \qquad \tilde{\rho}(\xi_2) = \mathcal{R}$$

where we drop subscripts.

2.5. Dual representations

We will now show how the elements of $\mathfrak{g}$ can be realized as operators on functions. Fix a basis $\{\xi_1, \ldots, \xi_d\}$ of $\mathfrak{g}$, with the corresponding basis $\{\,[\![n]\!]\,\}$ for $U(\mathfrak{g})$ (recall section 2.1). Then

$$g(A_1, \ldots, A_n; \xi_1, \ldots, \xi_d) = \exp(A_1\xi_1)\cdots\exp(A_d\xi_d) \tag{22}$$

for $A = (A_1, \ldots, A_d)$ in an appropriate neighbourhood of $0 \in \mathbb{R}^d$ defines a neighbourhood of the identity element of G. These coordinates are called coordinates of the second kind (see below, section 2.7, for coordinates of the first kind).

We interpret $g(A_1, \ldots, A_d; \xi_1, \ldots, \xi_d)$ as a formal pairing of bases between the algebra $\mathcal{A} = \boldsymbol{k}[A_1, \ldots, A_d]$ of polynomials in the commuting variables $A_1, \ldots, A_d$, and the universal enveloping algebra $U(\mathfrak{g})$.

For convenience, let A denote the d-tuple $(A_1, \ldots, A_d)$ and ξ denote $(\xi_1, \ldots, \xi_d)$.

For $\mathcal{A}$ choose the basis $\{c_n(A)\}$, where $c_n(A) = A^n/n! = (A_1^{n_1}/n_1!)\cdots(A_d^{n_d}/n_d!)$. Let $\mathcal{C}$ be the formal infinite-tuple with components $\{c_n(A)\}$ and Ψ denote the corresponding infinite-tuple $\{\,[\![n]\!]\,\}$. Then, expanding the exponentials in (22) shows that

$$g(A;\xi) = \sum_{n\in\mathbb{N}^d} c_n(A)[\![n]\!] = \langle \mathcal{C}, \Psi\rangle. \tag{23}$$

(We note that this interpretation is useful for generalizing our procedure to quantum groups, [2].)

On the group element $g(A;\xi)$, the action of the boson operators $\mathcal{R}_i$ and $\mathcal{V}_i$ transfers to operators acting on functions of the A-variables; i.e. shifting indices accordingly,

$$\mathcal{R}_i g(A;\xi) = \sum_n c_n(A)[\![n_1,\ldots,n_i+1,\ldots,n_d]\!] \tag{24}$$

$$= \frac{\partial}{\partial A_i} g(A;\xi) \tag{25}$$

$$\mathcal{V}_i g(A;\xi) = \sum_n c_n(A) n_i\, [\![n_1,\ldots,n_i-1,\ldots,n_d]\!] \tag{26}$$

$$= A_i\, g(A;\xi). \tag{27}$$

Now consider multiplying a basis element $[\![n]\!]$ by ξ_i. It results in a linear combination of the basis elements of the enveloping algebra. In general, we can write, using the canonical boson operators, $\xi_i[\![n]\!] = \sum_k \phi_{ik}(\mathcal{R})\pi_{ik}(\mathcal{V})[\![n]\!]$. Write this action, expressed in terms of bosons, as $\xi_i[\![n]\!] = \hat{\xi}_i[\![n]\!]$. Now, using the formal pairing, applying ξ_i term-by-term, we have

$$\xi_i g(A;\xi) = \langle \mathcal{C}, \xi_i\Psi\rangle = \langle \mathcal{C}, \hat{\xi}_i\Psi\rangle = \langle \xi_i^{\ddagger}\mathcal{C}, \Psi\rangle = \xi_i^{\ddagger} g(A;\xi)$$

i.e. we dualize the action on the enveloping algebra to that on functions of A. Similarly, multiplication on the right yields

$$g(A;\xi)\xi_i = \langle \mathcal{C}, \Psi\xi_i\rangle = \langle \xi_i^{\star}\mathcal{C}, \Psi\rangle = \xi_i^{\star} g(A;\xi)$$

(note that we consider the boson formulation $\hat{\xi}_i$ only for the left action). Now, since on g as a function of A the Lie algebra acts as vector fields, these operators are first order in the partial derivatives $\partial/\partial A_i$. Thus the action on the enveloping algebra, $\hat{\xi}_i$, is first order in the raising operators $\mathcal{R}_k$. Summarizing, we have (recall the abbreviated notation for partial derivatives with respect to the variables A_i, section 2.1):

Proposition 2.3.

(a) *The left multiplication by basis elements on the universal enveloping algebra* $U(\mathfrak{g})$*, dualizes to vector fields, the left dual representation; i.e. there is a matrix of functions,* $\pi^{\ddagger}(A)$ *such that*

$$\xi_i g(A;\xi) = \xi_i^{\ddagger} g(A;\xi) = \sum_j \pi_{ij}^{\ddagger}(A)\partial_j g(A;\xi).$$

This in turn is dual to the left action

$$\hat{\xi}_i = \sum_j \mathcal{R}_j \pi_{ij}^{\ddagger}(\mathcal{V})$$

of the action of $\mathfrak{g}$ *on* $U(\mathfrak{g})$ *expressed in terms of canonical bosons. This is called the double dual.*

(b) *The right multiplication by basis elements on the universal enveloping algebra* $U(\mathfrak{g})$*, dualizes to vector fields, the right dual representation; i.e. there is a matrix of functions,* $\pi^{\star}(A)$ *such that*

$$\xi_i g(A;\xi) = \xi_i^{\star} g(A;\xi) = \sum_j \pi_{ij}^{\star}(A)\partial_j g(A;\xi).$$

Thus maps $\rho^{\ddagger} : U(\mathfrak{g}) \to \mathrm{Hom}(\bar{\mathcal{A}}, \bar{\mathcal{A}})$ and $\rho^{*} : U(\mathfrak{g}) \to \mathrm{Hom}(\bar{\mathcal{A}}, \bar{\mathcal{A}})$ are induced, via $\rho^{\ddagger}(\xi_i) = \xi_i^{\ddagger}$ and $\rho^{*}(\xi_i) = \xi_i^{*}$ $\bar{\mathcal{A}}$ denoting smooth functions. They satisfy

$$\rho^{\dagger}(X_1) \circ \rho^{\dagger}(X_2) = \rho^{\dagger}(X_2 X_1) \tag{28}$$

$$\rho^{*}(X_1) \circ \rho^{*}(X_2) = \rho^{*}(X_1 X_2) \tag{29}$$

for $X_1, X_2 \in U(\mathfrak{g})$, i.e. ρ^* is a representation of $U(\mathfrak{g})$ and $\rho^{\dagger}$ is a representation of $U(\mathfrak{g})^{\mathrm{op}}$.

The double dual may be written using variables x_i and corresponding differential operators $D_i = \partial/\partial x_i$, via the replacements $\mathcal{R}_i \to x_i, \mathcal{V}_i \to D_i$. In this way, one can find representations on spaces of functions $f(x_1, \ldots, x_d)$ corresponding to quotient representations such as those found above from $U(\mathfrak{g})$, section 2.3.

2.5.1. Example: the affine group. For the affine group, define the group elements with respect to coordinates of the second kind by

$$g(A_1, A_2; \xi_1, \xi_2) = \exp(A_1\xi_1)\exp(A_2\xi_2). \tag{30}$$

This may be interpreted as a formal pairing of bases $\{c_{n_1n_2}(A) = (A_1^{n_1}/n_1!)(A_2^{n_2}/n_2!)\}$ and $\{[\![n]\!] = \xi_1^{n_1}\xi_2^{n_2}\}$. In practice, it is often convenient to obtain vector field realizations corresponding to left and right multiplication by using the adjoint representation of the group; i.e. the fact that the exponential of the adjoint representation in the Lie algebra is conjugation in the group. Using the commutation relation $[\xi_2, \xi_1] = \xi_2$, equivalently, $-(\mathrm{ad}\,\xi_1)(\xi_2) = \xi_2$, we have

$$\mathrm{e}^{-A_1 \mathrm{ad}\,\xi_1}\xi_2 = \xi_2 + A_1[-\xi_1, \xi_2] + \tfrac{1}{2}A_1^2[-\xi_1, [-\xi_1, \xi_2]] + \cdots = \mathrm{e}^{A_1}\xi_2 \tag{31}$$

$$\mathrm{e}^{A_2 \mathrm{ad}\,\xi_2}\xi_1 = \xi_1 + A_2[\xi_2, \xi_1] + \tfrac{1}{2}A_2^2[\xi_2, [\xi_2, \xi_1]] + \cdots = \xi_1 + A_2\xi_2. \tag{32}$$

Thus, for ξ_1 (with partial derivatives always with respect to the A-variables),

$$\xi_1^{\dagger} g = \xi_1 \mathrm{e}^{A_1\xi_1}\mathrm{e}^{A_2\xi_2} = \partial_1 g \tag{33}$$

$$\begin{aligned} \xi_1^{*} g &= \mathrm{e}^{A_1\xi_1}\mathrm{e}^{A_2\xi_2}\xi_1 = \mathrm{e}^{A_1\xi_1}\mathrm{e}^{A_2\xi_2}\xi_1\mathrm{e}^{-A_2\xi_2}\mathrm{e}^{A_2\xi_2} \\ &= \mathrm{e}^{A_1\xi_1}\mathrm{e}^{\mathrm{ad}\,A_2\xi_2}\xi_1\mathrm{e}^{A_2\xi_2} = \mathrm{e}^{A_1\xi_1}(\xi_1 + A_2\xi_2)\mathrm{e}^{A_2\xi_2} \\ &= (\partial_1 + A_2\partial_2)g \end{aligned} \tag{34}$$

and similarly for ξ_2.

Proposition 2.4. *The left dual representation is given by*

$$\xi_1^{\dagger} = \partial_1 \qquad \xi_2^{\dagger} = \mathrm{e}^{A_1}\partial_2$$

the right dual is

$$\xi_1^{*} = \partial_1 + A_2\partial_2 \qquad \xi_2^{*} = \partial_2$$

and the double dual (in $\{x, D\}$ variables) is

$$\hat{\xi}_1 = x_1 \qquad \hat{\xi}_2 = x_2\,\mathrm{e}^{D_1}.$$

The corresponding pi-matrices are

$$\pi^{\ddagger} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{A_1} \end{pmatrix} \qquad \pi^{\star} = \begin{pmatrix} 1 & A_2 \\ 0 & 1 \end{pmatrix}.$$

Note that the double dual expressed in $\{\mathcal{R}, \mathcal{V}\}$ variables is identical to the action of left multiplication on $U(\mathfrak{g})$ as we found above, equations (18) and (19).

Here are two quotient representations with bases and action written in terms of $\{x, D\}$ variables.

(a) For $\xi_1\Omega = \alpha\Omega$, the basis is $|m\rangle = \xi_2^m\Omega$ and

$$\hat{\xi}_1 = \alpha - x_2 D_2 \qquad \hat{\xi}_2 = x_2.$$

(b) For $\xi_2\Omega = \beta\Omega$, the basis is $|m\rangle = \xi_1^m\Omega$ and

$$\hat{\xi}_1 = x_1 \qquad \hat{\xi}_2 = \beta \mathrm{e}^{D_1}.$$

There are analogous representations for actions on the right.

2.5.2. Example: some nilpotent groups. The defining commutation relations imply, for $1 \leqslant j \leqslant N$,

$$\begin{aligned} \mathrm{e}^{A_0 \operatorname{ad} \xi_0}\xi_j &= \xi_j + A_0\xi_{j+1} + \frac{A_0^2}{2!}\xi_{j+2} + \cdots + \frac{A_0^{N-j}}{(N-j)!}\xi_N \\ &= \sum_{i=0}^{N-j} \frac{A_0^i}{i!}\xi_{j+i} \end{aligned} \tag{35}$$

$$\mathrm{e}^{-A_j \operatorname{ad} \xi_j}\xi_0 = \xi_0 + A_j\xi_{j+1}. \tag{36}$$

With

$$g(A;\xi) = g(A_1, \ldots, A_N, A_0; \xi_1, \ldots, \xi_N, \xi_0) = \mathrm{e}^{A_1\xi_1} \cdots \mathrm{e}^{A_N\xi_N}\mathrm{e}^{A_0\xi_0} \tag{37}$$

using the adjoint action we find

Proposition 2.5. *The left dual representation is given by*

$$\xi_0^\ddagger = \partial_0 + \sum_{i=1}^{N-1} A_i\partial_{i+1} \qquad \xi_j^\ddagger = \partial_j \qquad 1 \leqslant j \leqslant N$$

the right dual is

$$\xi_0^* = \partial_0 \qquad \xi_j^* = \sum_{i=0}^{N-j} \frac{A_0^i}{i!}\partial_{j+i} \qquad 1 \leqslant j \leqslant N$$

and the double dual (in $\{x, D\}$ variables) is

$$\hat{\xi}_0 = x_0 + \sum_{i=1}^{N-1} x_{i+1}D_i \qquad \hat{\xi}_j = x_j \qquad 1 \leqslant j \leqslant N.$$

For $N = 4$, the corresponding pi-matrices are

$$\pi^\ddagger = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & A_1 & A_2 & A_3 & 1 \end{pmatrix}$$

and

$$\pi^* = \begin{pmatrix} 1 & A_0 & A_0^2/2 & A_0^3/6 & 0 \\ 0 & 1 & A_0 & A_0^2/2 & 0 \\ 0 & 0 & 1 & A_0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which suggest the general pattern.

Remark 2.6. One can show that, in fact, $\pi^*(\pi^{\ddagger})^{-1}$ is the transpose of the group element $g(A; \check{\xi})$ formed by exponentiating the matrices of the adjoint representation of the Lie algebra (see [3, pp 36–7]).

2.6. *The composition law*

We will now look at a composition law for formal pairings. Since the formal pairings are local group elements this will be the group multiplication in coordinates of the second kind. Given two group elements $g(A; \xi)$ and $g(B; \xi)$ their product $g(A; \xi)g(B; \xi) = g(A \odot B; \xi)$ is also a group element. The notation $A \odot B$ indicates the product in terms of the coordinates. The composition law $\odot$ is, in general, uniquely defined only for a sufficiently small neighbourhood of 0. We give the explicit form of the composition law for our example groups (cf the group law as given in section 2.2).

2.6.1. *Example: the affine group.*

Proposition 2.7. *For the affine group, the composition law is*

$$A \odot B = (A_1 + B_1, e^{B_1} A_2 + B_2).$$

Proof. We want to calculate $e^{A_1\xi_1} e^{A_2\xi_2} e^{B_1\xi_1} e^{B_2\xi_2}$. The main question is how to commute the second term past the third one. Referring to equation (31), we have $e^{-B\xi_1}\xi_2 e^{B\xi_1} = e^B \xi_2$. Exponentiating both sides yields

$$e^{-B\xi_1} e^{A\xi_2} e^{B\xi_1} = \exp(e^B A\xi_2).$$

Thus,

$$e^{A_2\xi_2} e^{B_1\xi_1} = e^{B_1\xi_1} \exp(e^{B_1} A_2\xi_2)$$

and collecting terms yields the result. □

2.6.2. *Example: some nilpotent groups.* To multiply

$$e^{A_1\xi_1} \cdots e^{A_N\xi_N} e^{A_0\xi_0} e^{B_1\xi_1} \cdots e^{B_N\xi_N} e^{B_0\xi_0}$$

we have to commute $\exp(A_0\xi_0)$ past each $\exp(B_j\xi_j)$ for $1 \leqslant j \leqslant N$. From equation (35), we have, for $1 \leqslant k \leqslant N$, $e^{A_0\xi_0}\xi_k e^{-A_0\xi_0}$, and exponentiating yields

$$e^{A_0\xi_0} e^{B_k\xi_k} e^{-A_0\xi_0} = \exp\left(B_k \sum_{i=0}^{N-k} \frac{A_0^i}{i!}\xi_{k+i}\right).$$

Now set $k + i = j$ and recombine. We find $(A \odot B)_0 = A_0 + B_0$ and for $1 \leqslant j \leqslant N$,

$$(A \odot B)_j = A_j + \sum_{i=0}^{j-1} \frac{A_0^i}{i!} B_{j-i}.$$

We have thus recovered the group law, equation (7), in a slightly different notation.

It is easily seen that this approach is not feasible in more complicated situations. The next section shows how coordinates of the second kind and the composition law are related and provides a general approach to computations.

2.7. The splitting lemma

As a vector space with basis $\{\xi_1, \dots, \xi_d\}$, a typical element of $\mathfrak{g}$ has the form $X = \alpha_\mu \xi_\mu$ (here indices are summed, see section 2.1). Thus $\exp(X)$ is a group element in a neighbourhood of the identity. The α_i are local coordinates, the *coordinates of the first kind*. If we write the group element in terms of coordinates of the second kind, we have effectively factorized or *split* the exponential into a product of one-parameter subgroups. Thus the lemma relating the two types of coordinates is called the splitting lemma. This means finding the change-of-coordinates mapping $\alpha \to A(\alpha)$; i.e. we are interested in the relation

$$g = \mathrm{e}^X = g(A; \xi) = \mathrm{e}^{A_1(\alpha)\xi_1} \cdots \mathrm{e}^{A_d(\alpha)\xi_d}.$$

(We continue to use A to denote the coordinates of the second kind and $A(\alpha)$ to denote the mapping from the α coordinates.) In fact, the factorization corresponds to the right and left dual vector fields and the flow of the group (composition) law. To see this, consider the left dual:

$$X\, g(A; \xi) = X^{\ddagger} g(A; \xi) = \alpha_\lambda \pi^{\ddagger}_{\lambda\mu} \partial_\mu\, g(A; \xi).$$

Denote $\alpha = (\alpha_1, \dots, \alpha_d)$, so that $t\alpha = (t\alpha_1, \dots, t\alpha_d)$ for a real parameter t. Let $x(t) = A(t\alpha) \odot A$ denote the 'flow of the group law', for t in some neighbourhood of 0. For any smooth function, the chain rule says that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)) = \dot{x}_\mu \frac{\partial f}{\partial x_\mu}$$

where the dot denotes differentiation with respect to t.

Now consider $g(x(t); \xi) = g(A(t\alpha); \xi) g(A; \xi) = \mathrm{e}^{tX} g(A; \xi)$. Differentiating with respect to t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x(t); \xi) = X \mathrm{e}^{tX}\, g(A; \xi) = X^{\ddagger} g(x(t); \xi). \tag{38}$$

Since, in $g(x(t); \xi)$, the coordinates $x_i = x_i(t)$ replace the A variables, the left dual acts with evaluations and differentiations with respect to the x-variables; i.e.

$$\dot{x}_\mu \partial_\mu g(x(t); \xi) = \alpha_\lambda \pi^{\ddagger}_{\lambda\mu}(x(t)) \partial_\mu\, g(x(t); \xi).$$

Thus,

$$\dot{x}_i = \alpha_\lambda \pi^{\ddagger}_{\lambda i}(x).$$

A similar argument, pulling down X as in equation (38), to the right of g, yields the corresponding result for $x(t) = A \odot A(t\alpha)$. So,

Lemma 2.8. Flow of the group. *Let $X = \alpha_\mu \xi_\mu$. Let $A(\alpha)$ be the map of coordinates determined by*

$$\exp(X) = g(A; \xi) = \mathrm{e}^{A_1(\alpha)\xi_1} \cdots \mathrm{e}^{A_d(\alpha)\xi_d}.$$

Let $\odot$ denote the group law: $g(A; \xi)g(B; \xi) = g(A \odot B; \xi)$.

(a) Let $x(t) = A(t\alpha) \odot A$. Then $x(t)$ satisfies the equations $\dot{x}_j = \alpha_\lambda \pi^{\ddagger}_{\lambda j}(x)$, with the initial condition $x(0) = A$.
(b) Let $x(t) = A \odot A(t\alpha)$. Then $x(t)$ satisfies the equations $\dot{x}_j = \alpha_\lambda \pi^{*}_{\lambda j}(x)$, with the initial condition $x(0) = A$.

We may reformulate this in terms of vector fields.

Corollary 2.9.

(a) The integral curves of the vector field $X^{\ddagger} = \alpha_\lambda \pi^{\ddagger}_{\lambda\mu}(A)\partial_\mu$ are of the form $A(t\alpha) \odot A$.
(b) The integral curves of the vector field $X^{*} = \alpha_\lambda \pi^{*}_{\lambda\mu}(A)\partial_\mu$ are of the form $A \odot A(t\alpha)$.

Now follows

Lemma 2.10 (Splitting lemma). *Let $X = \alpha_\mu \xi_\mu$. Consider the factorization*

$$\exp(X) = g(A; \xi) = \mathrm{e}^{A_1(\alpha)\xi_1} \cdots \mathrm{e}^{A_d(\alpha)\xi_d}.$$

Let $\tilde{\pi}$ denote the coefficient matrix (pi-matrix) of either the left or the right dual representation, Then the coordinate map $\alpha \to (A_1(\alpha), \ldots, A_d(\alpha))$ is determined as follows. Solve the differential equations

$$\dot{x}_j = \alpha_\lambda \tilde{\pi}_{\lambda j}(x) \qquad j = 1, \ldots, d \tag{39}$$

with the initial conditions $x_1(0) = \cdots = x_d(0) = 0$. Then $A_i(\alpha) = x_i(1)$, for $1 \leqslant i \leqslant d$.

Proof. From lemma 2.8, for $\tilde{\pi} = \pi^{\ddagger}$, we have $x(1) = A(\alpha) \odot A$. With the initial variables $A_i = 0$, $1 \leqslant i \leqslant d$, we have $x(1) = A(\alpha)$ as required. Note that for $\tilde{\pi} = \pi^{*}$, the zero initial conditions yield the same result. □

An interesting corollary is

Corollary 2.11. *For the coordinate map $\alpha \to A$ of coordinates of the first kind to coordinates of the second kind, we have the identity*

$$\alpha_\lambda \pi^{\ddagger}_{\lambda j}(A(\alpha)) = \alpha_\lambda \pi^{*}_{\lambda j}(A(\alpha))$$

for $1 \leqslant j \leqslant d$.

Remark 2.12. By remark 2.6, taking transposes, this may be reformulated as $\check{\pi}(A(\alpha))\alpha = \alpha$, where $\check{\pi}$ is the group element formed by exponentiating the adjoint representation; i.e. this shows invariance of the α's under the adjoint group.

Example 2.13 (the affine group). Using the pi-matrices from proposition 2.4, we have for the left flow $\dot{x}_1 = \alpha_1$, $\dot{x}_2 = \alpha_2 \mathrm{e}^{x_2}$ with the solution

$$x_1(t) = A_1 + \alpha_1 t \qquad x_2(t) = A_2 + \mathrm{e}^{A_1}(\alpha_2/\alpha_1)(\mathrm{e}^{\alpha_1 t} - 1).$$

For the right flow, we have $\dot{x}_1 = \alpha_1$, $\dot{x}_2 = \alpha_1 x_2 + \alpha_2$ with the solution

$$x_1(t) = A_1 + \alpha_1 t \qquad x_2(t) = (\alpha_2/\alpha_1)(\mathrm{e}^{\alpha_1 t} - 1) + A_2 \mathrm{e}^{\alpha_1 t}.$$

Now, setting $A_i = 0$, $i = 1, 2$, yields $A(t\alpha)$, and, furthermore, setting $t = 1$, we have the coordinate map

$$A_1(\alpha) = \alpha_1 \qquad A_2(\alpha) = (\alpha_2/\alpha_1)(\mathrm{e}^{\alpha_1} - 1)$$

and from this we can check the consistency with the flow of the group, lemma 2.8, via proposition 2.7.

Similar results may be found for the nilpotent groups using the pi-matrices, for example, see after proposition 2.5.

2.7.1. Left dual flow. Since the right dual map $\xi_i \to \xi_i^*$ induces a homomorphism of Lie algebras, we have, from corollary 2.9, with $\xi_i^* = \pi^*(A)_{i\mu}\partial_\mu$,

$$g(B; \xi^*) f(A) = f(A \odot B) \tag{40}$$

for smooth functions f. However, the left dual $\xi_i \to \xi_i^\ddagger$ induces an antihomomorphism. By composing with the map S, $S(X) = -X$, we have that $\xi_i \to -\xi_i^\ddagger$ induces a homomorphism of Lie algebras. Thus, we have,

$$\exp(\alpha_\mu(-\xi_\mu^\ddagger)) = \exp(-A_1(\alpha)\xi_1^\ddagger) \cdots \exp(-A_d(\alpha)\xi_d^\ddagger).$$

Now take inverses (in the group) to obtain

$$\exp(\alpha_\mu \xi_\mu^\ddagger) = \exp(A_d(\alpha)\xi_d^\ddagger) \cdots \exp(A_1(\alpha)\xi_1^\ddagger)$$

i.e. we have the 'opposite group'. We denote the group element with the basis ordered in reverse by $\overleftarrow{g}$. Since the coordinate map $\alpha \to A$ is locally invertible, we conclude that $\overleftarrow{g}(A; \xi^\ddagger)$ is a general group element in a neighbourhood of the identity. Now multiply

$$\exp(\beta_\mu(-\xi_\mu^\ddagger)) \exp(\alpha_\mu(-\xi_\mu^\ddagger)) = g(B; -\xi^\ddagger) g(A; -\xi^\ddagger) = g(B \odot A; -\xi^\ddagger).$$

Again, taking inverses, we have

$$\overleftarrow{g}(A; \xi^\ddagger) \overleftarrow{g}(B; \xi^\ddagger) = \overleftarrow{g}(B \odot A; \xi^\ddagger).$$

As expected, the order of composition is reversed.

In analogy to equation (40), from corollary 2.9 we have the action

$$\overleftarrow{g}(B; \xi^\ddagger) f(A) = f(B \odot A). \tag{41}$$

2.8. Matrix elements

Exponentiating the representation ρ_L of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ we obtain a representation of G on $U(\mathfrak{g})$. To simplify notation we will identify $\rho_L(\xi_j)$ and ξ_j when acting on $U(\mathfrak{g})$. We define the matrix elements $\left\langle {m \atop n} \right\rangle_A$ of the representation G on $U(\mathfrak{g})$ by

$$g(A; \xi)[\![n]\!] = \sum_m \left\langle {m \atop n} \right\rangle_A [\![m]\!] \tag{42}$$

where $[\![n]\!] = \xi_1^{n_1} \cdots \xi_d^{n_d}$ is the Poincaré–Birkhoff–Witt basis of $U(\mathfrak{g})$, noted in section 2.1.

These matrix elements are types of special functions and typically can be expressed in terms of generalized hypergeometric functions. For a complete discussion of how their special function properties can be derived see [3, 4].

The following proposition gives a useful formula for calculating the matrix elements.

Proposition 2.14 (Principal formula). *With the standard basis* $c_m(A) = (A_1^{m_1}/m_1!)\cdots(A_d^{m_d}/m_d!)$ *for polynomials in A, the matrix elements are given by*

$$\left\langle {m \atop n} \right\rangle_A = (\xi^*)^n c_m(A) \tag{43}$$

where $(\xi^*)^n = (\xi_1^*)^{n_1}\cdots(\xi_d^*)^{n_d}$*, basis monomials in terms of the right dual representation.*

Proof. Write the product of group elements $g(A;\xi)$ and $g(B;\xi)$ as

$$\begin{aligned} g(A;\xi)g(B;\xi) &= g(A,\xi)\sum_n c_n(B)[\![n]\!] \\ &= \sum_n c_n(B)g(A;\xi)[\![n]\!] \\ &= \sum_{m,n} c_n(B)\left\langle {m \atop n} \right\rangle [\![m]\!] \end{aligned} \tag{44}$$

since the A's and B's commute. On the other hand, since the right dual gives a homomorphism of Lie algebras, we can also write, cf equation (40),

$$\begin{aligned} g(A;\xi)g(B;\xi) &= g(B;\xi^*)g(A;\xi) \\ &= \sum_{n,m} c_n(B)(\xi^*)^n c_m(A)[\![m]\!]. \end{aligned} \tag{45}$$

Comparing these two expressions leads to the desired formula. □

We mention some of the many interesting relations for the matrix elements that can now be deduced from the group law and the relations of the operators ξ^*. This approach to special functions is in the spirit of the classic work of Vilenkin (see Klimyk and Vilenkin [8]).

Addition theorems. Writing the group law (as in the above proof)

$$g(A;\xi)g(B;\xi) = \sum_{m,n} c_n(B)\left\langle {m \atop n} \right\rangle_A [\![m]\!] \tag{46}$$

and as

$$g(A\odot B;\xi) = \sum_m c_m(A\odot B)[\![m]\!] \tag{47}$$

we read off the transformation formula

$$c_m(A\odot B) = \sum_n \left\langle {m \atop n} \right\rangle_A c_n(B) \tag{48}$$

that is, the coefficients c_n transform as a vector for the representation. Similarly,

$$g(A;\xi)g(B;\xi)[\![n]\!] = g(A\odot B;\xi)[\![n]\!] \tag{49}$$

yields the addition theorem

$$\left\langle {m \atop n} \right\rangle_{A\odot B} = \left\langle {m \atop \lambda} \right\rangle_A \left\langle {\lambda \atop n} \right\rangle_B \tag{50}$$

where in the implied summation λ is a multi-index. So these are indeed a matrix representation of the group acting on $U(\mathfrak{g})$.

Differential recurrence relations. Since the right dual representation gives a homomorphism of Lie algebras, we have

$$\begin{aligned}\xi_i^* \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A &= \xi_i^* (\xi^*)^n c_m(A) \\ &= \sum_r M_{rn}(\xi_i)(\xi^*)^r c_m(A) \\ &= \sum_r \left\langle \begin{matrix} m \\ r \end{matrix} \right\rangle_A M_{rn}(\xi_i) \end{aligned} \tag{51}$$

where $M_{mn}(\xi_i)$ are the matrix elements of $\rho_L(\xi_i)$, as in equation (10). Recall that this action is the same as the double dual $\hat{\xi}_i = \mathcal{R}_\mu \pi^{\ddagger}_{i\mu}(\mathcal{V})$ acting on the n-indices. In other words,

$$\xi_i^* \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A = \hat{\xi}_i \left\langle \begin{matrix} m \\ \boldsymbol{n} \end{matrix} \right\rangle_A$$

with the boldface indicating that the multi-index n is varied.

2.8.1. Example: the affine group. The matrix elements are given by the principal formula (proposition 2.14)

$$\left\langle \begin{matrix} m_1 m_2 \\ n_1 n_2 \end{matrix} \right\rangle_{(A_1,A_2)} = (\xi_1^*)^{n_1} (\xi_2^*)^{n_2} (A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!). \tag{52}$$

Using the right dual as given in proposition 2.4 gives us

$$\begin{aligned}\left\langle \begin{matrix} m_1 m_2 \\ n_1 n_2 \end{matrix} \right\rangle_{(A_1,A_2)} &= (\partial_1 + A_2 \partial_2)^{n_1} (\partial_2)^{n_2} (A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!) \\ &= \binom{n_1}{\mu} \partial_1^{n_1-\mu} (A_2\partial_2)^\mu \partial_2^{n_2} (A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!) \\ &= \binom{n_1}{\mu} \frac{A_1^{m_1-n_1+\mu}}{(m_1-n_1+\mu)!} (m_2-n_2)^\mu \frac{A_2^{m_2-n_2}}{(m_2-n_2)!}. \end{aligned} \tag{53}$$

Let ${}_1F_1\left(\begin{smallmatrix} a \\ b \end{smallmatrix} \,\middle|\, x\right) = \sum_{k=0}^{\infty} [(a)_k x^k/(b)_k k!]$, using standard notation for general hypergeometric functions. For $m_1 \geqslant n_1$, the summation is readily converted to a ${}_1F_1$ function. If $m_1 < n_1$, a change of summation index $\mu = \lambda - (m_1 - n_1)$ leads to the following.

Proposition 2.15. *Let $\Delta = m - n = (m_1 - n_1, m_2 - n_2)$. For the affine group, the matrix elements are given by*

$$\left\langle \begin{matrix} m_1 m_2 \\ n_1 n_2 \end{matrix} \right\rangle_{(A_1,A_2)} = \begin{cases} \dfrac{A_1^{\Delta_1}}{\Delta_1!} \dfrac{A_2^{\Delta_2}}{\Delta_2!} \, {}_1F_1\left(\begin{matrix} -n_1 \\ \Delta_1 + 1 \end{matrix} \,\middle|\, -A_1 \Delta_2 \right) & \textit{if} \quad \Delta_1 \geqslant 0 \quad \Delta_2 \geqslant 0 \\ \dbinom{n_1}{-\Delta_1} \Delta_2^{-\Delta_1} \dfrac{A_2^{\Delta_2}}{\Delta_2!} \, {}_1F_1\left(\begin{matrix} -m_1 \\ -\Delta_1 + 1 \end{matrix} \,\middle|\, -A_1 \Delta_2 \right) & \textit{if} \quad \Delta_1 < 0 \quad \Delta_2 \geqslant 0. \end{cases}$$

Bringing in the double dual, $\hat{\xi}_1 = \mathcal{R}_1$, $\hat{\xi}_2 = \mathcal{R}_2 \, \mathrm{e}^{\mathcal{V}_1}$, we find the following differential recurrence relations:

$$\begin{aligned} (\partial_1 + A_2 \partial_2) \left\langle \begin{matrix} m_1 m_2 \\ n_1 n_2 \end{matrix} \right\rangle_{(A_1,A_2)} &= \left\langle \begin{matrix} m_1, m_2 \\ n_1 + 1, n_2 \end{matrix} \right\rangle_{(A_1,A_2)} \\ \partial_2 \left\langle \begin{matrix} m_1 m_2 \\ n_1 n_2 \end{matrix} \right\rangle_{(A_1,A_2)} &= \sum_{0 \leqslant k \leqslant n_1} \binom{n_1}{k} \left\langle \begin{matrix} m_1, m_2 \\ n_1 - k, n_2 + 1 \end{matrix} \right\rangle_{(A_1,A_2)}. \end{aligned}$$

These may be rewritten using proposition 2.15 to yield identities and recursion formulae for the ${}_1F_1$ function.

Remark 2.16.

(a) Similar results may be found for the nilpotent groups using the dual representations found above.
(b) One can find recurrence relations not involving derivatives that generalize the well known 'contiguous relations' satisfied by general hypergeometric functions. See the references cited above for this and further results; see, e.g., [4].

3. Evolution equations and processes on a Lie group

Consider the equation

$$\frac{\partial u}{\partial t} = H(D)u \tag{54}$$

for $u(x, t)$ with initial condition $u(x, 0) = f(x)$. Assume the operator $H(D)$ is given as follows. Let $H(z)$ be analytic and defined in a neighbourhood of the origin in the complex plane by

$$\mathrm{e}^{tH(z)} = \int_{-\infty}^{\infty} \mathrm{e}^{zx}\, p_t(\mathrm{d}x) \tag{55}$$

where p_t is a convolution family of probability measures on the real line. H is called an *analytic generator*. Then the solution to equation (54) is given by

$$u(x, t) = \int_{-\infty}^{\infty} f(x + y)\, p_t(\mathrm{d}y)$$

for bounded continuous functions f and for polynomials f.

If in $H(D_1, \ldots, D_d)$ we (can) replace (unambiguously) each D_i by ξ_i, a chosen basis for the Lie algebra $\mathfrak{g}$, then we write $H(\xi)$. A typical case is where $H(z) = H_1(z_1) + \cdots + H_d(z_d)$ is a sum of analytic generators H_i each depending only on one variable. Consider the equation $u_t = H(\xi)u$. This equation is subject to various interpretations. We mention two. Namely,

- We replace ξ by an operator realization of the Lie algebra, for example, by vector fields such as ξ^*, and find solutions $u(A, t)$ with polynomial initial condition $u(A, 0) = f(A)$. In this sense, corollary 2.9 solves the case where H is linear, i.e. either $H = \alpha_\mu \xi^*_\mu$ or $H = \alpha_\mu \xi^\ddagger_\mu$.
- Write the equation in operator form: $u_t = [H, u]$. We look for solutions $u(\xi, t)$ with initial conditions $u(\xi, 0)$ in $U(\mathfrak{g})$.

When H is given according to equation (55) there is a stochastic process (Lévy process or a process with stationary, independent increments) on $\mathbb{R}^d$ with a distribution at each fixed t given by the probability measure p_t. Let $y(t)$ denote this stochastic process starting from the origin. We map it into the Lie group in the following way. For a time increment $\Delta_i t = t_i - t_{i-1}$, let $\Delta_i y = y(t_i) - y(t_{i-1})$, for $i = 1, \ldots, N$. Define the multiplicative increment

$$g(\Delta_i y; \xi) = \mathrm{e}^{(\Delta_i y)_1 \xi_1} \cdots \mathrm{e}^{(\Delta_i y)_d \xi_d}.$$

Now consider the product integral over the time interval $[0, t]$

$$\prod g(\Delta y; \xi) = g(\Delta_1 y; \xi) g(\Delta_2 y; \xi) \cdots g(\Delta_N y; \xi)$$

with time increasing from left to right. We have a process $Y(t)$ on the group if $\Delta t = \sup_i \Delta_i t \to 0$ as $N \to \infty$ and the product converges. Then we write the product integral

$$g(Y(t); \xi) = \lim_{\Delta t \to 0} \prod g(\Delta y; \xi) = \prod g(\mathrm{d}y; \xi)$$

cf McKean [10], Hakim-Dowek and Lépingle [7].

One can also consider discrete-time processes, in which case no limit is necessary.

Remark 3.1. This approach also works for quantum groups (see [2]).

For Y, we have the following integral formula.

Theorem 3.2. *The process on the group satisfies*

$$Y(t) = \int_0^t \big(Y(s) \odot \mathrm{d}y - Y(s)\big)$$

in the sense that the corresponding group element exists (as a product integral) if and only if this integral exists.

Proof. Write $\Delta_i t = t_i - t_{i-1}$. Then

$$g(Y(t_{i-1} + \Delta_i t); \xi) = g(Y(t_{i-1}); \xi) g(\Delta_i y; \xi) = g(Y(t_{i-1}) \odot \Delta_i y; \xi)$$

so that $Y(t_i) = Y(t_{i-1} + \Delta_i t) = Y(t_{i-1}) \odot \Delta_i y$. Now subtract $Y(t_{i-1})$ and integrate, i.e. summing and letting $\Delta t \to 0$ yields the result. □

Note that this gives an Itô, non-anticipating, stochastic integral.

Now consider the case where the components y_i are mutually independent. Then we have, with angled brackets denoting the expected value,

$$\langle \exp(z_\mu y_\mu(t)) \rangle = \mathrm{e}^{t H_1(z_1)} \cdots \mathrm{e}^{t H_d(z_d)}.$$

For a multiplicative increment, we have

$$\begin{aligned} \langle g(\Delta_i y; \xi) \rangle &= \langle \mathrm{e}^{(\Delta_i y)_1 \xi_1} \cdots \mathrm{e}^{(\Delta_i y)_d \xi_d} \rangle \\ &= \mathrm{e}^{(\Delta_i t) H_1(\xi_1)} \cdots \mathrm{e}^{(\Delta_i t) H_d(\xi_d)} \end{aligned}$$

where we can integrate each factor separately by independence. Now we have

$$\left\langle \prod g(\Delta y; \xi) \right\rangle = \prod \mathrm{e}^{(\Delta_i t) H_1(\xi_1)} \cdots \mathrm{e}^{(\Delta_i t) H_d(\xi_d)}.$$

Here from the Trotter product formula we find on the right-hand side the sum of the generators even in the non-commutative case; e.g., take each $\Delta_i t$ to be t/N for fixed t. Then we have

$$\left\langle \prod g(\Delta y; \xi) \right\rangle = \big(\mathrm{e}^{(t/N) H_1(\xi_1)} \cdots \mathrm{e}^{(t/N) H_d(\xi_d)}\big)^N$$

i.e.

Proposition 3.3. *Let* $y(t) = (y_1(t), \ldots, y_d(t))$ *have independent components with* $\langle \exp(z_i y_i(t)) \rangle = \exp(t H_i(z_i))$. *Then the corresponding process on the group satisfies*

$$\langle g(Y(t); \xi) \rangle = \exp\big[t\big(H_1(\xi_1) + \cdots + H_d(\xi_d)\big)\big].$$

3.1. Left dual process

If instead of $g(A;\xi)$ or $g(A;\xi^*)$ we use $\overleftarrow{g}(A;\xi^\ddagger)$ in the above construction, cf section 2.7.1, the $\odot$ products involved are reversed, i.e. the process is built up from right to left:

$$\overleftarrow{g}(\Delta_1 y;\xi^\ddagger)\cdots\overleftarrow{g}(\Delta_N y;\xi^\ddagger) = \overleftarrow{g}(\Delta_N y\odot\cdots\odot\Delta_1 y;\xi^\ddagger)$$

theorem 3.2 becomes

Theorem 3.4. *The left dual process on the group satisfies*

$$Y(t) = \int_0^t \big(\mathrm{d}y\odot Y(s) - Y(s)\big)$$

in the sense that the corresponding group element exists (as a product integral) if and only if this integral exists.

What is interesting is that proposition 3.3 is virtually unchanged. The reason is that in the limit the generator becomes the sum of the individual generators and thus is symmetric with respect to reversal; i.e.

Proposition 3.5. *Let* $y(t) = (y_1(t),\ldots,y_d(t))$ *have independent components with* $\langle\exp(z_i y_i(t))\rangle = \exp(tH_i(z_i))$. *Then the corresponding left dual process on the group satisfies*

$$\langle\overleftarrow{g}(Y(t);\xi^\ddagger)\rangle = \exp\big[t\big(H_1(\xi_1^\ddagger)+\cdots+H_d(\xi_d^\ddagger)\big)\big].$$

Example 3.6. Let us illustrate this with the affine group. Here an interesting phenomenon occurs that arises due to the time asymmetry of the construction. In the direct construction, the composition law $A\odot B = (A_1+B_1, \mathrm{e}^{B_1}A_2+B_2)$ causes difficulties since the second component looks like $\int_0^t(\mathrm{e}^{\mathrm{d}y_1}-1)Y_2(s)+\mathrm{d}y_2$. However, if we use the left dual construction, we find that the process takes the form $Y(t) = \big(y_1(t), \int_0^t \mathrm{e}^{y_1(s)}\,\mathrm{d}y_2(s)\big)$.

4. Appell systems and evolution equations

Appell systems $\{h_n(x); n\in\mathbb{N}\}$ on $\mathbb{R}$ are classically characterized by the two conditions:

- $h_n(x)$ is a polynomial of degree n;
- $\frac{\mathrm{d}}{\mathrm{d}x}h_n(x) = nh_{n-1}(x)$.

Interesting examples are furnished by the shifted moment sequences

$$h_n(x) = \int_{-\infty}^{\infty}(x+y)^n\,p(\mathrm{d}y) \tag{56}$$

where p is a probability measure on $\mathbb{R}$ with all moments finite. In the case where $H(z)$ is given as in equation (55), this is the solution to the evolution equation $u_t = H(D)u$ with $u(x,0) = x^n$ at $t=1$. With a complex change of variables, this includes the Hermite polynomials

$$H_n(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x+\mathrm{i}y)^n\mathrm{e}^{-y^2/2}\,\mathrm{d}y \tag{57}$$

for the Gaussian case. Here we wish to use the basic construction of [5] to define Appell systems on Lie groups and consider some explicit examples.

These Appell systems are of interest since they provide (analogues of) polynomial solutions of evolution equations on Lie groups, including natural generalizations of the classical heat equation on Euclidean spaces.

Taking our cue from formula (56), let Y denote a random variable on $\mathbb{R}^d$.

Definition 4.1. *The left and right Appell systems (corresponding to the random variable Y) on the Lie group are defined as*

$$h_n^L(A) = \langle c_n(Y \odot A)\rangle \qquad h_n^R(A) = \langle c_n(A \odot Y)\rangle \tag{58}$$

with the angle brackets denoting the expected value.

Using the fact that c_n transforms as a vector, i.e. $c_n(A \odot B) = \sum_m \left\langle {n \atop m} \right\rangle_A c_m(B)$ we have

Proposition 4.2. *The left and right Appell systems satisfy*

$$h_n^L(A) = \sum_m c_m(A) \left\langle \left\langle {n \atop m} \right\rangle_Y \right\rangle \qquad h_n^R(A) = \sum_m \left\langle {n \atop m} \right\rangle_A \langle c_m(Y)\rangle.$$

Of particular interest are Appell systems related to the processes $Y(t)$ constructed in the previous section. We explicitly indicate t dependence in h_n.

Proposition 4.3.

(a) Let $Y(t)$ satisfy $\langle g(Y(t); \xi)\rangle = \mathrm{e}^{tH(\xi)}$. Then

$$h_n^R(A, t) = \mathrm{e}^{tH(\xi^*)} c_n(A).$$

Thus, $u(A, t) = h_n^R(A, t)$ satisfies the evolution equation $u_t = H(\xi^*)u$ with the initial condition $u(A, 0) = c_n(A)$.

(b) Let $Y(t)$ be a left dual process, satisfying

$$\langle \overleftarrow{g}(Y(t); \xi^\ddagger)\rangle = \mathrm{e}^{tH(\xi^\ddagger)}.$$

Then

$$h_n^L(A, t) = \mathrm{e}^{tH(\xi^\ddagger)} c_n(A).$$

Thus, $u(A, t) = h_n^L(A, t)$ satisfies the evolution equation $u_t = H(\xi^\ddagger)u$ with the initial condition $u(A, 0) = c_n(A)$.

Proof. Apply equations (40) and (41) with $f(A) = c_n(A)$. Then the results follow from the definitions. □

5. Examples

5.1. Heat polynomials for the affine group

In this section we study the Appell systems solving some heat equations related to the affine group.

It is convenient to take a realization satisfying the commutation relations $[\xi_2, \xi_1] = \xi_1$. Using the machinery developed above, one finds the right dual $\xi_1^* = \mathrm{e}^{A_2}\partial_1$, $\xi_2^* = \partial_2$. Using the right dual (for example) and equation (40), we find the group law

$$A \odot B = (A_1 + \mathrm{e}^{A_2} B_1, A_2 + B_2).$$

One realization of this algebra is given by the operators on functions of one variable: $\xi_1 = -\mathrm{i}D$, $\xi_2 = -(xD + a)$, for some constant a, here $\mathrm{i} = \sqrt{-1}$, $D = \mathrm{d}/\mathrm{d}x$ as usual. Take $y(t) = (w_1(t), w_2(t))$, independent standard Wiener processes and denote the process on the group by $W(t)$. That is, $H(z) = \frac{1}{2}(z_1^2 + z_2^2)$, corresponding to the standard Laplacian on $\mathbb{R}^2$. From theorem 3.2, we have

$$W_1(t) = \int_0^t \mathrm{e}^{w_2(s)}\,\mathrm{d}w_1(s) \qquad W_2(t) = w_2(t).$$

For polynomial f (say),

$$\mathrm{e}^{tH(\xi)} f(x) = \langle \mathrm{e}^{\xi_1 W_1(t)} \mathrm{e}^{\xi_2 W_2(t)} f(x)\rangle = \langle \mathrm{e}^{-aW_2(t)} f(\mathrm{e}^{-W_2(t)}(x - \mathrm{i}W_1(t))\rangle$$

with $H(\xi) = \frac{1}{2}\left((xD + a)^2 - D^2\right)$. The eigenfunctions of H are Gegenbauer polynomials, satisfying

$$((xD + a)^2 - D^2) C_n^a(x) = (n + a)^2 C_n^a(x)$$

with

$$C_n^a(x) = \sum_k \frac{(a)_{n-k}}{(n - 2k)!k!}(-1)^k (2x)^{n-2k} \tag{59}$$

$$x^n = \frac{n!\,\Gamma(a)}{2^n} \sum_k \frac{a + n - 2k}{k!(a + n - k)!} C_{n-2k}^a(x). \tag{60}$$

Therefore, taking $f(x) = x^n/n!$, we have the Appell systems

$$\langle \mathrm{e}^{-aW_2(t)} \mathrm{e}^{-nW_2(t)} (x - \mathrm{i}W_1(t))^n\rangle = \frac{\Gamma(a)}{2^n} \sum_k \frac{a + n - 2k}{k!(a + n - k)!} C_{n-2k}^a(x)\,\mathrm{e}^{t(n+a-2k)^2/2}$$

satisfying $u_t = H(\xi)\,u$. An interesting special case arises taking $a = 0$. Then the eigenfunctions of H are Chebyshev polynomials T_n:

$$((xD)^2 - D^2) T_n = n^2 T_n$$

with

$$T_n(x) = \sum_k \binom{n-k}{k} \frac{n/2}{n - k} (-1)^k (2x)^{n-2k} \tag{61}$$

$$x^n = 2^{-n} \sum_k \binom{n}{k} T_{n-2k}(x). \tag{62}$$

Therefore, with $f(x) = x^n$,

$$\langle \mathrm{e}^{-nW_2(t)} (x - \mathrm{i}W_1(t))^n\rangle = 2^{-n} \sum_k \binom{n}{k} \mathrm{e}^{t(n-2k)^2/2} T_{n-2k}(x). \tag{63}$$

Now consider $H(\xi^*) = \frac{1}{2}\left(\mathrm{e}^{2x_2} D_1^2 + D_2^2\right)$, writing the right dual in $\{x, D\}$ variables. Then, via the group law,

$$\exp(t(H(\xi^*)) f(x) = \langle g(W; \xi^*) f(x)\rangle = \langle f(x_1 + \mathrm{e}^{x_2} W_1(t), x_2 + W_2(t))\rangle.$$

Choosing $f(x) = x_1^n \mathrm{e}^{-nx_2}$, suggested by equation (63),

$$\begin{aligned}\exp(t(H(\xi^*)) f(x) &= \langle (x_1 + \mathrm{e}^{x_2} W_1(t))^n \mathrm{e}^{-n(x_2+W_2(t))}\rangle \\ &= \langle (x_1 \mathrm{e}^{-x_2} + W_1(t))^n \mathrm{e}^{-nW_2(t)}\rangle.\end{aligned}$$

Denoting the right-hand side of equation (63) by $\phi_n(x)$, comparing with the above equation shows that

$$\langle (x_1 \mathrm{e}^{-x_2} + W_1(t))^n \mathrm{e}^{-nW_2(t)}\rangle = \mathrm{i}^n \phi_n(-\mathrm{i}x_1 \mathrm{e}^{-x_2})$$

i.e. we have the Lie reduction

$$v = x_1 e^{-x_2} \qquad \left(e^{2x_2} D_1^2 + D_2^2\right) f(v) = \left(\left(v\frac{d}{dv}\right)^2 + \frac{d^2}{dv^2}\right) f(v).$$

This reduction depends only on group properties, and in this context shows the equivalence of the corresponding classical processes.

5.2. *Heat polynomials on some nilpotent groups*

5.2.1. Heisenberg group. First take the $N = 2$ case of the class of nilpotent groups we have been considering. Take the realization

$$\xi_1 = x \qquad \xi_2 = 1 \qquad \xi_0 = D.$$

For the process, take any $y_1(t)$ of the type we have been considering, independent of $w_0(t)$, which we take to be a standard Wiener process. Take $y_2(t) = 0$. Thus, $H(\xi) = \frac{1}{2}D^2 + H_1(x)$. From the group law, section 2.6.2, $A \odot B = (A_1+B_1, A_2+B_2+A_0B_1, A_0+B_0)$ and theorem 3.2, we have

$$Y(t) = \left(y(t), \int_0^t w(s)\,dy(s), w(t)\right).$$

Thus,

$$\begin{aligned}\langle g(Y;\xi) f(x)\rangle &= \left\langle e^{xy(t)} \exp\left(\int_0^t w(s)\,dy(s)\right) e^{w(t)D} f(x)\right\rangle \\ &= \left\langle \exp\left(\int_0^t (x + w(s))\,dy(s)\right) f(x + w(t))\right\rangle.\end{aligned}$$

Integrating over the $y(t)$ process, by independence, we have

$$\exp\left[t\left(\tfrac{1}{2}D^2 + H_1(x)\right)\right] f(x) = \left\langle \exp\left(\int_0^t H_1(x + w(s))\,ds\right) f(x + w(t))\right\rangle$$

i.e. we have recovered the classical Feynman–Kac formula. For computational aspects of the Feynman–Kac formula, see, for example, Korzenowski [9].

For the right dual, in $\{x, D\}$ variables, from proposition 2.5 we have $H(\xi^*) = \frac{1}{2}D_0^2 + H_1(D_1 + x_0 D_2)$. Here take $y_2(t) = 0$, with $y_0(t) = w_0(t)$, $y_1(t) = w_1(t)$, independent standard Wiener processes, denoting the process on the group by $W(t)$. Then via the group law it follows that

$$\begin{aligned}\langle g(W;\xi^*) f(x)\rangle &= \langle f(x \odot W)\rangle \\ &= \left\langle f(x_1 + w_1(t), x_2 + \int_0^t (x_0 + w_0(s))\,dw_1(s), x_0 + w_0(t))\right\rangle.\end{aligned}$$

Taking f to be of the special form $f(x_1, x_2, x_0) = e^{x_2}\phi(x_0)$ shows that the Feynman–Kac formula is related to the three-dimensional process by a Lie reduction; i.e. the choice of a special combination of variables.

Note that the generator of the process $H = \frac{1}{2}(D_0^2 + (D_1 + x_0 D_2)^2)$ acts nilpotently on polynomials; i.e. there are finite-dimensional invariant polynomial subspaces and the operator exponential e^{tH} reduces to a finite series. Thus, moments of the process, i.e. joint correlation functions of the components, up to any given order can be computed, either symbolically or by choosing a polynomial basis and converting the operator to a matrix.

5.2.2. Higher-order nilpotent case. Here we continue with the case $N = 3$ of the nilpotent examples. This will suggest how the theory goes for general N. We write the details for the Wiener process, i.e. take $y_0(t) = w_0(t)$, $y_1(t) = w_1(t)$, independent standard Wiener processes, as above, with $y_2(t) = y_3(t) = 0$, for simplicity.

First, take the realization

$$\xi_1 = \tfrac{1}{2}x^2 \qquad \xi_2 = x \qquad \xi_3 = 1 \qquad \xi_0 = D.$$

Then, with $H = \frac{1}{2}(D_0^2 + \frac{1}{4}x^4)$ we have the group law

$$A \odot B = (A_1 + B_1, A_2 + B_2 + A_0 B_1, A_3 + B_3 + A_0 B_2 + (\tfrac{1}{2}A_0^2)B_1, A_0 + B_0)$$

and the process

$$\begin{aligned} W(t) &= (W_1(t), W_2(t), W_3(t), W_0(t)) \\ &= \left(w_1(t), \int_0^t w_0(s)\,\mathrm{d}w_1(s), \tfrac{1}{2}\int_0^t w_0(s)^2\,\mathrm{d}w_1(s), w_0(t) \right). \end{aligned}$$

Thus, with W_2 and W_3 given in the above equation, we have

$$\langle g(W;\xi) f(x)\rangle = \langle \mathrm{e}^{x^2 w_1(t)/2 + x W_2(t) + W_3(t)} f(x_0 + w_0(t))\rangle.$$

In this form the moments are not accessible, nor are there polynomial solutions.

Now, take the right dual, in $\{x, D\}$ variables. Then

$$H = \tfrac{1}{2}\big(D_0^2 + \big(D_1 + x_0 D_2 + (\tfrac{1}{2}x_0^2)D_3\big)^2\big)$$

and

$$\langle g(W;\xi^*) f(x)\rangle = \langle f(x_1 + w_1, x_2 + W_2 + x_0 w_1, x_3 + W_3 + x_0 W_2 + (\tfrac{1}{2}x_0^2)w_1, x_0 + w_0)\rangle.$$

Now the generator acts nilpotently, we have polynomial solutions, and moments to any given order can be calculated. Finally, note we have a Lie reduction to the one-variable case similar to that for $N = 2$.

6. Conclusion

The interplay of Lie theory—Lie algebras, Lie groups, the theory of Lie symmetries of differential equations—and the theory of stochastic processes is a subject of great interest. Much remains to be done. In the mathematical physics context, this area ties in with path integrals, including methods involving changes of variables and reduction methods for path integrals. Our approach to computing with Lie algebras and Lie groups, surveyed in the first part of this work, provides effective tools for working in this context. This provides a mathematical foundation for doing symbolic computations in the modern computing environment, using systems such as *Maple* and *Mathematica* that provide support for exact calculations.

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